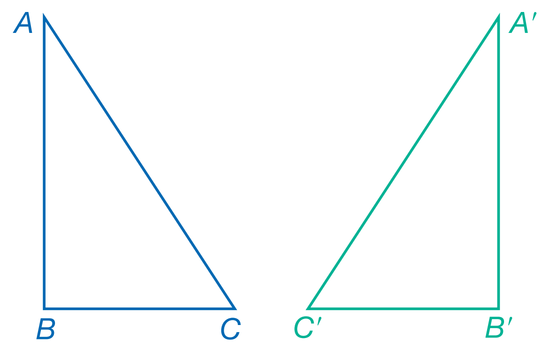
LESSON 6

Congruent Triangles

Congruence with Multiple Sides

UNDERSTAND Two triangles are congruent if all of their corresponding angles are congruent and all of their corresponding sides are congruent. However, you do not need to know the measures of every side and angle to show that two triangles are congruent.

$\triangle ABC$ is formed from three line segments: $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Suppose those segments were pulled apart and used to build another triangle, such as $\triangle A'B'C'$ shown. This new triangle could be formed by reflecting $\triangle ABC$ over a vertical line. A reflection is a rigid motion, so $\triangle A'B'C'$ must be congruent to $\triangle ABC$. In fact, any triangle built with these segments could be produced by performing rigid motions on $\triangle ABC$, so all such triangles would be congruent to $\triangle ABC$.



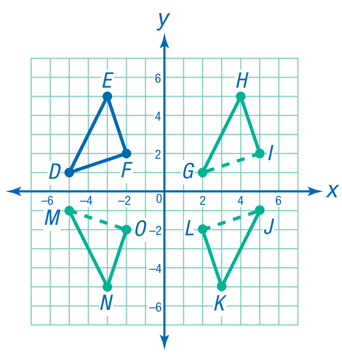
So, knowing all of the side lengths of two triangles is enough information to determine if they are congruent.

Side-Side-Side (SSS) Postulate: If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

UNDERSTAND You can also use the Side-Angle-Side (SAS) Postulate to prove that two triangles are congruent.

Side-Angle-Side (SAS) Postulate: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

Look at $\triangle DEF$ on the coordinate plane. Applying rigid motions to two of its sides produced new line segments. Sides $\overline{DE}$ and $\overline{EF}$ were translated 7 units to the right to form $\overline{GH}$ and $\overline{HI}$. $\overline{DE}$ and $\overline{EF}$ were rotated 180° about the origin to form $\overline{JK}$ and $\overline{KL}$. $\overline{DE}$ and $\overline{EF}$ were reflected over the x-axis to form $\overline{MN}$ and $\overline{NO}$.

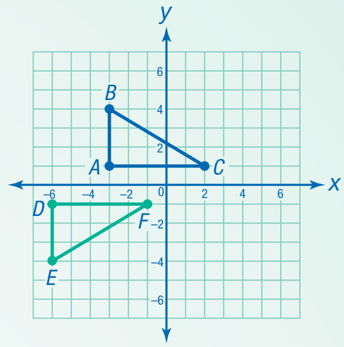


Each rigid motion preserved the lengths of the segments as well as the angle between them. In all three images, only one segment can be drawn to complete each triangle. Those line segments, shown as dotted lines, are congruent to $\overline{DF}$. So, $\triangle DEF \cong \triangle GHI \cong \triangle JKL \cong \triangle MNO$. The symbol $\cong$ means "is congruent to."

Connect

The coordinate plane on the right shows $\triangle ABC$ and $\triangle DEF$.

Use the SAS Postulate to show that the triangles are congruent. Then, identify rigid motions that could transform $\triangle ABC$ into $\triangle DEF$.



1 Make a plan.

Each triangle has one horizontal side and one vertical side that intersect, so both are right triangles. Since all right angles are congruent, the triangles have at least one pair of corresponding congruent angles. To prove the triangles congruent by the SAS Postulate, find the lengths of the adjacent sides (legs) that form the right angles.

2 Find and compare the lengths of the corresponding legs.

Count the units to find the length of each leg.

$AB = 3 \quad DE = 3$

$AC = 5 \quad DF = 5$

$\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$

▶ The triangles have two pairs of corresponding congruent sides, and their included angles are congruent right angles. So, according to the SAS Postulate, $\triangle ABC \cong \triangle DEF$.

3 Identify rigid motions that can transform $\triangle ABC$ to $\triangle DEF$.

Study the shapes of the triangles.

▶ $\overline{AC}$ corresponds to $\overline{DF}$ and is parallel to it. Vertex B is above $\overline{AC}$ on the left, while vertex E lies below $\overline{DF}$ and is also on the left. $\triangle ABC$ could be reflected over the x-axis. After such a reflection, $\triangle A'B'C'$ would have vertices $A'(-3, -1)$, $B'(-3, -4)$, and $C'(2, -1)$. To transform this image to $\triangle DEF$, translate it 3 units to the left.

DISCUSS

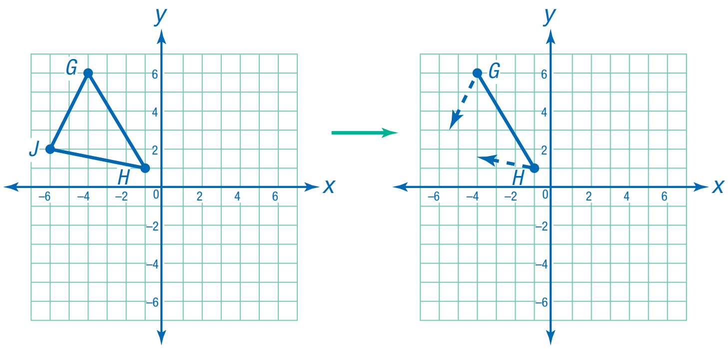
If $\triangle ABC$ had instead been rotated 90° and then translated down 1 unit, would the resulting image be congruent to $\triangle ABC$? How could you prove your answer?

Congruence with Multiple Angles

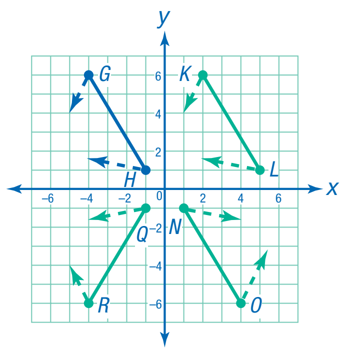
UNDERSTAND A third method for proving that triangles are congruent is the Angle-Side-Angle Theorem.

Angle-Side-Angle (ASA) Theorem: If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

Look at the coordinate planes below. Triangle *GHJ* is shown on the left. On the right, sides $\overline{HJ}$ and $\overline{GJ}$ of the triangle have been replaced by rays.

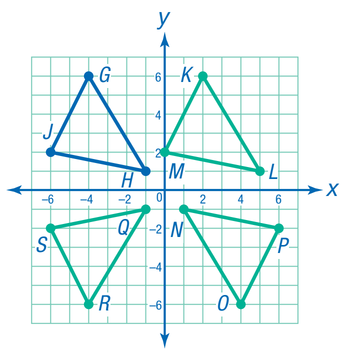


The coordinate plane on the right of this paragraph shows three transformations of $\overline{GH}$ and the rays extending from points *G* and *H*. $\overline{KL}$ is a translation of $\overline{GH}$ 6 units to the right. $\overline{ON}$ is a 180° rotation of $\overline{GH}$. $\overline{RQ}$ is a reflection of $\overline{GH}$ over the *x*-axis.



Each of those rigid motions has carried the angle-side-angle combination to a new location. The segments $\overline{KL}$, $\overline{ON}$, and $\overline{RQ}$ are congruent to $\overline{GH}$. Rigid motions also preserved the angles formed by the segment and each ray.

The coordinate plane on the lower right shows the triangles formed by extending the rays until they intersect.

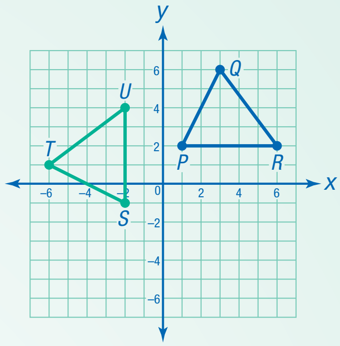


In each case, the rays can only intersect at one point and thus can form only one triangle. Each of these triangles is congruent to $\triangle GHJ$.

Connect

Triangles *PQR* and *STU* are shown on the coordinate plane on the right.

Given that $\angle P \cong \angle S$ and $\angle R \cong \angle U$, prove that the triangles are congruent. Then identify rigid motions that can transform $\triangle PQR$ into $\triangle STU$.



1 Make a plan.

You already know that two pairs of corresponding angles are congruent. If you can show that the included sides are the same length, then the triangles are congruent by the ASA Theorem.

2 Find the lengths of the included sides.

In $\triangle PQR$, the included side of $\angle P$ and $\angle R$ is $\overline{PR}$. Find the length of $\overline{PR}$ by counting units.

$PR = 5$

In $\triangle STU$, the included side for $\angle S$ and $\angle U$ is $\overline{SU}$. Find the length of $\overline{SU}$ by counting units.

$SU = 5$

▶ Since two pairs of corresponding angles and the included sides are congruent, $\triangle PQR \cong \triangle STU$ by the ASA Theorem.

3 Identify rigid motions that could transform $\triangle PQR$ into $\triangle STU$.

Study the shapes of the triangles.

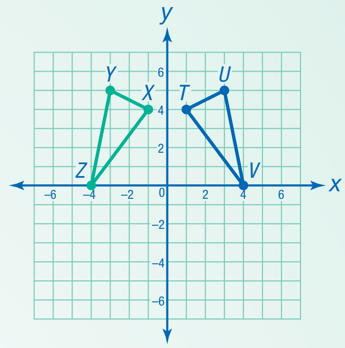
Side $\overline{PR}$ is horizontal, and corresponding side $\overline{SU}$ is vertical. It appears that $\triangle PQR$ was rotated 90° counterclockwise. If that rotation were around the origin, $\triangle P'Q'R'$ would have vertices at $P'(-2, 1)$, $Q'(-6, 3)$, and $R'(-2, 6)$. To transform this image to $\triangle STU$, translate it 2 units down.

▶ Triangle *STU* can be produced by rotating $\triangle PQR$ 90° counterclockwise and then translating the image down 2 units.

DISCUSS

In triangles *ABC* and *DEF*, $\angle A = \angle D = 20°$, $\angle B = \angle E = 60°$, and $\angle C = \angle F = 100°$. Are triangles *ABC* and *DEF* congruent?

EXAMPLE A Show that $\triangle TUV$ is congruent to $\triangle XYZ$.



1 Make a plan.

To use the SSS Postulate, you need to show that each side on $\triangle XYZ$ has a corresponding congruent side on $\triangle TUV$. To do so, show that each side of $\triangle XYZ$ is a rigid-motion transformation of a side of $\triangle TUV$.

2 Compare the endpoints of $\overline{XY}$ and $\overline{TU}$.

Points T and X have the same y-coordinates but opposite x-coordinates. The same is true for points U and Y. This indicates that $\overline{TU}$ can be reflected over the y-axis to form $\overline{XY}$. As a result, you know that $\overline{XY} \cong \overline{TU}$.

3 Follow the same process for the other two pairs of corresponding sides.

The endpoints of $\overline{YZ}$ and $\overline{UV}$ have the same y-coordinates but opposite x-coordinates.

The endpoints of $\overline{XZ}$ and $\overline{TV}$ also have the same y-coordinates but opposite x-coordinates.

So, $\overline{YZ}$ is a reflection of $\overline{UV}$ over the y-axis, and $\overline{XZ}$ is a reflection of $\overline{TV}$ over the y-axis.

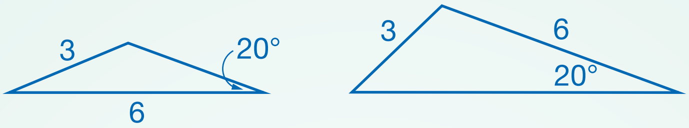
4 Draw a conclusion.

▶ Since each side of $\triangle XYZ$ is a reflection of the corresponding side of $\triangle TUV$, the corresponding sides of the triangles are congruent. According to the SSS Postulate, $\triangle TUV \cong \triangle XYZ$.

DISCUSS

How does $\angle Z$ compare to $\angle V$? Could one angle have a greater measure than the other?

EXAMPLE B Ian is studying wing designs for airplanes. He compares two wings whose triangular cross sections both contain one 20° angle, an adjacent side that measures 6 feet, and a non-adjacent side that measures 3 feet. Determine if the triangular cross sections are identical.

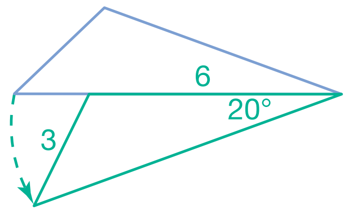


1 Make a plan.

Identical triangles are congruent, so rigid motions should carry one of the figures onto the other. Transform one of the figures so that known congruent parts line up, and compare the other parts.

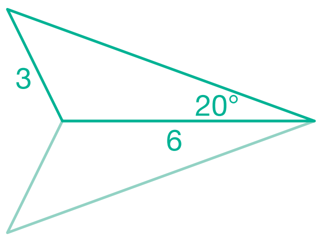
2 Attempt to align the angle and adjacent side.

The angles are already aligned, but the 6-foot sides are not. Rotate the second triangle 20° counterclockwise so that its 6-foot side is also horizontal.

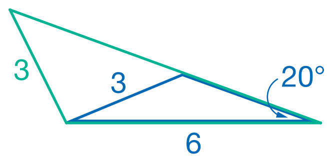


3 Re-align the angles.

The angles are no longer aligned, so reflect the image vertically to bring them back into alignment.



4 Translate the image onto the other triangle.



The angle and adjacent side are aligned, but the remaining sides and angles do not match.

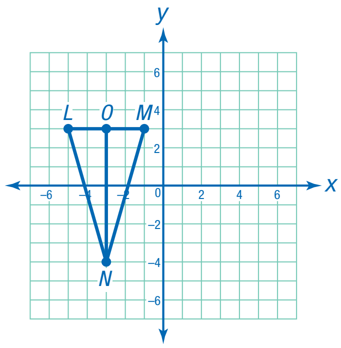
▶ The cross sections are not congruent.

DISCUSS

If two triangles have two pairs of congruent sides and one pair of congruent angles, can you prove that the triangles are congruent?

Practice

Use the coordinate plane below for questions 1–4.



1. $\angle LON$ and $\angle MON$ both measure ____________ degrees.

2. $LO = OM =$ ____________ units

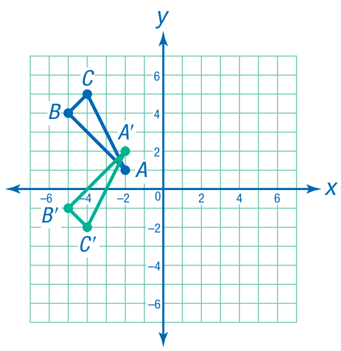
3. $\triangle LON$ and $\triangle MON$ share side ____________.

4. Triangles *LON* and *MON* are congruent by the ____________ Postulate.

HINT If two triangles share a side, that line segment is the same length in both triangles.

Choose the best answer.

5. Which pair of rigid motions shows that $\triangle ABC$ and $\triangle A'B'C'$ are congruent?



A. reflection across the *x*-axis followed by a translation of 3 units up

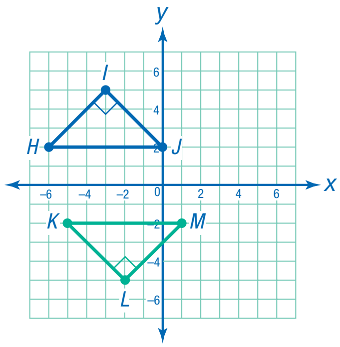
B. reflection across the *x*-axis followed by a translation of 3 units down

C. rotation of 180° about the origin followed by a reflection over the *y*-axis

D. rotation of 90° counterclockwise about the origin followed by a translation of 4 units down

6. The coordinate plane on the right shows isosceles right triangles *HIJ* and *KLM*.

Use the ASA Postulate to prove that $\triangle HIJ$ and $\triangle KLM$ are congruent. Identify rigid motions that could transform $\triangle HIJ$ into $\triangle KLM$.



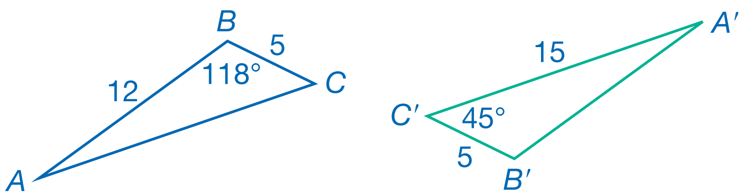
7. Triangle *ABC* was reflected horizontally, reflected vertically, and then translated to form triangle $A'B'C'$. Identify the lengths and angle measures below.

$A'B' =$ __________

$AC =$ __________

$m\angle B' =$ __________

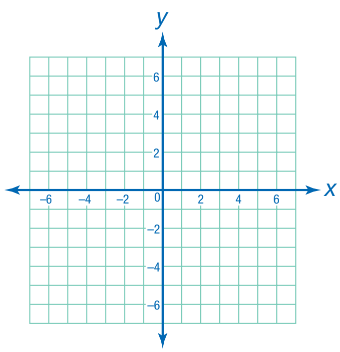
$m\angle A =$ __________



Use the following information for questions 8 and 9.

Right triangle *NOP* has vertices *N*(1, 1), *O*(1, 5), and *P*(4, 1).
Right triangle *QRS* has vertices *Q*(−4, −1), *R*(−4, −5), and *S*(−1, −1).

8. SKETCH Sketch both triangles on the coordinate plane.



9. PROVE Prove that $\triangle NOP$ and $\triangle QRS$ are congruent.